245a. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

PCP', DCD' are conjugate diameters of an ellipse; PN, DM are the ordinates to the major axis at P and D; show $CM/PN=CN/DM=AC/BC$, and that AP and BD' are parallel, and that AP' is parallel to BD.

I. Solution by J. SCHEFFER, Kee Mar College, Hagerstown, Md.

Denote AC by a, BC by b, CP by a', CD by b', $\angle PCA$ by ϕ, and $\angle DCM$ by λ. From the well known relations of conjugate diameters of an ellipse, we have

$$\sin^2\phi=\frac{b^2(a^2-a_1^2)}{a_1^2(a^2-b^2)},\quad \cos^2\phi=\frac{a^2(a_1^2-b^2)}{a_1^2(a^2-b^2)},$$

$$\sin^2\lambda=\frac{b^2(a^2-b_1^2)}{b_1^2(a^2-b^2)},\quad \cos^2\lambda=\frac{a^2(b_1^2-b^2)}{b_1^2(a^2-b^2)}.$$

$$\therefore CM=b_1\cos\lambda=a\sqrt{\frac{b_1^2-b^2}{a^2-b^2}},\quad CN=a_1\cos\phi=a\sqrt{\frac{a_1^2-b^2}{a^2-b^2}},$$

$$PN=a_1\sin\phi=b\sqrt{\frac{a^2-a_1^2}{a^2-b^2}},\quad DM=b_1\sin\lambda=b\sqrt{\frac{a^2-b_1^2}{a^2-b^2}}.$$

$$\therefore \frac{CM}{PN}=\frac{a}{b}\sqrt{\frac{b_1^2-b^2}{a^2-a_1^2}}=\frac{a}{b},\ \text{since } a_1^2+b_1^2=a^2+b^2,\ \text{and}$$

$$\frac{CN}{DM}=\frac{a}{b}\sqrt{\frac{a_1^2-b^2}{a^2-b_1^2}}=\frac{a}{b}.$$

The coördinates of the point P are $\left(a\sqrt{\dfrac{a_1^2-b^2}{a^2-b^2}},\ b\sqrt{\dfrac{a^2-a_1^2}{a^2-b^2}}\right)$, and of A, $(a, 0)$; those of B are $(0, b)$, and of D' $\left(a\sqrt{\dfrac{b_1^2-b^2}{a^2-b^2}},\ -b\sqrt{\dfrac{a^2-b_1^2}{a^2-b^2}}\right)$; therefore

$$\text{slope of } AP=b\sqrt{\frac{a^2-a_1^2}{a^2-b^2}}\div a\left(\sqrt{\frac{a_1^2-b^2}{a^2-b^2}}-1\right)=-\frac{b}{a}\frac{\sqrt{(a_1^2-b^2)}+\sqrt{(a^2-b^2)}}{\sqrt{(a^2-a_1^2)}}.$$

$$\text{Slope of } BD'=-\frac{b}{a}\cdot\frac{1+\sqrt{\left(\dfrac{a^2-b_1^2}{a^2-b^2}\right)}}{\sqrt{\left(\dfrac{b_1^2-b^2}{a^2-b^2}\right)}}=-\frac{b}{a}\cdot\frac{\sqrt{(a^2-b_1^2)}+\sqrt{(a^2-b^2)}}{\sqrt{(b_1^2-b)}}.$$

By the relation $a_1^2+b_1^2=a^2+b^2$, both slopes are equal; therefore PA is paralle to BD'. In the same way, AP' is parallel to BD.

II. Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

Since P and D are extremities of conjugate diameters, their coördinates may be written $(a\cos\phi,\ b\sin\phi)$, $(a\sin\phi,\ -b\cos\phi)$ where a and b are the semi-axes. We obtain the required result at once by disregarding the sign and considering only the absolute magnitudes of the coördinates. Equations to AP and BD are

$$\frac{x-a\cos\phi}{a-a\cos\phi}=\frac{y-b\sin\phi}{-b\sin\phi},\ \text{and}\ \frac{x-a\sin\phi}{-a\sin\phi}=\frac{y+b\cos\phi}{b+b\cos\phi},$$

and the inclinations are therefore

$$-\frac{b\sin\phi}{a(1-\cos\phi)}\ \text{and}\ -\frac{b(1+\cos\phi)}{a\sin\phi},$$

which are easily seen to be equal.

Also solved by G. B. M. Zerr.

CALCULUS.

187. Proposed by L. T. JACKSON, St. Louis, Mo.

Find the area of the ellipse

$$x=a_1+a_2\cos\theta+a_3\sin\theta,$$
$$y=b_1+b_2\cos\theta+b_3\sin\theta.$$

I. Solution by H. B. LEONARD, B. S., Chicago, Ill.

Let $x'=x-a_1$, $y'=y-b_1$; then eliminating $\cos\theta$ and $\sin\theta$ in turn we obtain

$$b_2x'-a_2y'=\triangle\sin\theta,\ b_3x'-a_3y'=-\triangle\cos\theta,$$

where $\triangle=b_2a_3-a_2b_3$. The substitution $x''=b_2x'-a_2y'$, $y''=b_3x'-a_3y'$, which multiplies areas by $\triangle$, transforms the ellipse into the circle $x''=\triangle\sin\theta$, $y''=-\triangle\cos\theta$; *i. e.* $x''^2+y''^2=\triangle^2$.

The area of circle being $\pi\triangle^2$, the area of ellipse is $\pi\triangle=\pi(b_2a_3-a_2b_3)$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Transform the origin to the center by writing x for $x-a_1$, y for $y-b_1$.

$\therefore x=a_2\cos\theta+a_3\sin\theta$, $y=b_2\cos\theta+b_3\sin\theta$.

$$\text{Area}=\int ydx=\int_0^{2\pi}(b_2\cos\theta+b_3\sin\theta)(a_3\cos\theta-a_2\sin\theta)d\theta$$

$$=\int_0^{2\pi}(a_3b_2\cos^2\theta-a_2b_3\sin^2\theta-a_2b_2\sin\theta\cos\theta\pm a_3b_3\sin\theta\cos\theta)d\theta$$

$$=\pi(a_3b_2-a_2b_3).$$

Also solved by F. P. Matz, and the Proposer.